ELSEVIER Journal of Pure and Applied Algebra 119 (1997) 53–65

JOURNAL OF PURE AND APPLIED ALGEBRA

The fibre of an iterative adjunction of cells

Yves Félix *

Université Catholique de Louvain, 2, Chemin du Cyclotron, 1348 Louvain-La-Neuve, Belgium

Communicated by J.D. Stasheff; received 3 April 1995; revised 19 September 1995

Abstract

Let (Y,X) be a relative CW complex with X and Y simply-connected and suppose that the relative homology $\mathrm{H}_*(Y,X;k)$ is nonzero. Denote by F the homotopy fibre of the inclusion $X \hookrightarrow Y$. We show that the grade of $\mathrm{H}_*(F;k)$ as a module over $\mathrm{H}_*(\Omega Y;k)$ is less than the relative cone length $\mathrm{cl}(Y,X)$. This result appears as a corollary of a deeper result concerning differential modules over differential graded algebras. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 55P35, 55P62

1. Introduction

Let X be a simply connected space and let Y be obtained by attaching to X a wedge of cells along a continuous map

$$\vee h_i : \vee S^{n_i} \longrightarrow X,$$

where the n_i are positive integers. Each map h_i factors up to homotopy through the homotopy fibre F of the inclusion $i : X \to Y$:

$$h_i \sim j \circ g_i : \qquad S^{n_i} \xrightarrow{g_i} F \xrightarrow{j} X.$$

We put $a_i = \mathrm{H}_{n_i}(g_i)(\alpha_i)$ where α_i is a generator of $\mathrm{H}_{n_i}(S^{n_i};\mathbb{Z})$. In [9], Thomas and the author prove that the integral reduced homology of F, $\tilde{\mathrm{H}}_*(F;\mathbb{Z})$, is a free $\mathrm{H}_*(\Omega Y;\mathbb{Z})$-module generated by the elements a_i.

We consider here a more general situation. Let (Y,X) be a relative CW complex where X and Y are simply connected CW complexes. The *relative cone length* $\mathrm{cl}(Y,X)$, (Y,X) is the integer defined by the inequality $\mathrm{cl}(Y,X) \leq n$ if and only if there is a

* E-mail: felix@agel.ucl.ac.be.

PII S0022-4049(96)00018-7

sequence of cofibrations

$$E_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1} \cong C_{f_i}, \qquad i = 0, \dots, n-1,$$

with $X_0 \cong X$, $X_n \cong Y$, and where the spaces E_i are wedges of spheres. By taking X to be a point, we recover the absolute version of cone length [3].

More generally, for a continuous map between simply connected CW complexes, $f : M \to N$, we write $\mathrm{cl}(N,M)$ for the relative cone length of the inclusion of M into the mapping cylinder of f.

Let (Y,X) be a relative CW complex. We denote by F the homotopy fibre of the inclusion $X \hookrightarrow Y$. Our first result concerns the structure of the reduced homology $\tilde{\mathrm{H}}_*(F;k)$ of F as a module over $\mathrm{H}_*(\Omega Y;k)$ when k is a field.

Theorem 1. *With the previous notations, if the relative homology* $\mathrm{H}_*(Y,X;k)$ *is nonzero, then*

$$\mathrm{grade}_{\mathrm{H}_*(\Omega Y;k)}\tilde{\mathrm{H}}_*(F;k) \le \mathrm{cl}(Y,X) - 1.$$

Moreover in case of equality, the homological dimension of $\tilde{\mathrm{H}}_*(F;k)$ *as an* $\mathrm{H}_*(\Omega Y;k)$*-module is equal to* $\mathrm{cl}(Y,X) - 1$.

To be self-contained, we recall that the grade of a graded module M over a graded k-augmented algebra A is the least integer p such that the graded $\mathrm{Ext}^p_A(M,A)$ is nonzero. The depth of the graded algebra A is by definition the grade of the trivial module k. Theorem 1 has to be considered in relation with the following theorem of Halperin, Thomas and the author concerning the grade of the whole homology of the fibre F.

Theorem (Félix et al. [6]). *In a fibration* $F \to E \xrightarrow{p} B$, *we have*

$$\mathrm{grade}_{\mathrm{H}_*(\Omega B;k)}\mathrm{H}_*(F;k) \le \mathrm{cat}\, p\,.$$

Since the grade of a finite-dimensional nonzero module over a graded augmented algebra A is equal to the depth of this algebra, Theorem 1 has the following corollary.

Corollary. *Let* $F \to E \to B$ *be a fibration such that the dimension of* $\tilde{\mathrm{H}}_*(F;k)$ *is finite and* $\neq 0$; *then* $\mathrm{cl}(B,E) > \mathrm{depth}\,\mathrm{H}_*(\Omega B;k)$.

Theorem 1 is in fact a corollary of a deeper result concerning differential modules over differential graded algebras. Let us begin with some definitions. Let (A, d_A) be a k-augmented chain algebra over a field k.

Definition. (i) A *semi-free* differential A-module C is a graded differential right A-module with a filtration, $C = \bigcup_{r\ge 0} C(r)$, compatible with the differential, $d(C(r)) \subset C(r)$, and such that the different quotients $C(r)/C(r-1)$ are free differential A-modules:

$$C(r)/C(r-1) \cong (V_r \otimes A, 1 \otimes d_A)\,.$$

(ii) A semi-free differential A-module of the form $C = \bigcup_{r=0}^{n-1} C(r)$ is called an *n-stage* differential A-module.

The semi-free modules have some important properties [5], [8]. Recall first that two morphisms of differential A-modules $f, g : (M,d) \to (N,d)$ of degree p are homotopic, $f \sim g$, if there is a A-linear map of degree $p+1$, $h : M \to N$ such that $f - g = dh + (-1)^{|p|}hd$. We then have:

Proposition 1 (Félix et al. [8]). *Let (M,d) be a semi-free differential A-module and $(M,d) \xrightarrow{g} (N,d) \xleftarrow{f} (P,d)$ be morphisms of differential A-modules. If* $\mathrm{H}(f)$ *is an isomorphism, then there is a morphism of differential A-modules $q : (M,d) \to (P,d)$ such that $fq \sim g$.*

Proposition 2 (Félix et al. [8]). *Let $f : (M,d) \to (N,d)$ be a morphism of semi-free differential A-modules inducing an isomorphism in homology, then there is a morphism of A-modules $g : (N,d) \to (M,d)$ such that fg and gf are respectively homotopic to the identity on N and M.*

In this paper we prove:

Theorem 2. *Let C be a finite type n-stage differential A-module. If C is not acyclic, then* $\mathrm{grade}_{\mathrm{H}(A)}\mathrm{H}(C) < n$.

Moreover if $\mathrm{grade}_{\mathrm{H}(A)}\mathrm{H}(C) = n-1$, *then* $\mathrm{H}(C)$ *is an* $\mathrm{H}(A)$*-module of homological dimension* $n-1$.

This theorem gives a strong property concerning the homology of a differential module: If E is an $\mathrm{H}(A)$-module of grade q, then E cannot be the homology of an n-stage differential A-module (and in particular of an n-stage differential $\mathrm{H}(A)$-module) with $n \le q$.

We have to let remark that the gradation is important for the homology but has less importance for the grade. For instance, if $A = k[x]/x^5$, $|x| = 2$, and if $H = ky \oplus kz$ is a trivial A-module, $|y| = |z| = 2$, then the grade of H is 0, but the semi-free differential A-modules C with $\mathrm{H}(C) = H$ are ∞-stage. On the other hand, the trivial A-module $H' = ky' \oplus kz$, $|z| = 2, |y'| = 13$, is a grade 0 module and is the homology of the 1-stage differential module C', $C'_0 = z.k[x]/x^5$, $C'_1 = t.k[x]/x^5$, $D(t) = z.x$.

Let us come back to the iterative adjunction of cells. When $\mathrm{cl}(Y,X) = 1$, Theorem 1 implies that $\tilde{\mathrm{H}}_*(F;k)$ is a free $H_*(\Omega Y;k)$-module, and we recover the initial result of Thomas and the author [9]. When $\mathrm{cl}(Y,X) = 2$, the grade of $\tilde{\mathrm{H}}_*(F;k)$ is less than or equal to 1. If the grade is zero, we have a nontrivial morphism of $\mathrm{H}_*(\Omega Y;k)$-modules $\varphi : \tilde{\mathrm{H}}_*(F;k) \to \mathrm{H}_*(\Omega Y;k)$. In the case the loop space homology $\mathrm{H}_*(\Omega Y;k)$ has an exponential growth, this implies that the homology $\mathrm{H}_*(F;k)$ has also an exponential growth. When the grade is one and $\mathrm{H}_*(Y,X;k)$ is finite dimensional, we have an equality of formal series $\sum_n \dim \tilde{\mathrm{H}}_n(F;k)t^n = P(t)(\sum_n \dim \mathrm{H}_n(\Omega Y;k)t^n)$, where $P(t)$ is a polynomial.

In general, only a little is known about the relative behaviour of $H_*(F;k)$ and $H_*(\Omega Y;k)$ in a fibration $F \to X \to Y$. For instance, they are not necessarily rationally related. Recall that two series $A(t)$ and $B(t)$ are rationally related if there are polynomials $P(t), R(t), S(t)$ and $Q(t)$ such that $A(t) = (P(t)\cdot B(t) + Q(t))/(R(t)\cdot B(t) + S(t))$, with $P(t)S(t) \neq Q(t)R(t)$.

Consider for instance a 2-cone $Y = (\bigvee_{i\in I} S^{n_i}) \cup \left(\bigcup_{j\in J} e^{m_j}\right)$ and the fibration

$$S^3 \to X = S^3 \times Y \xrightarrow{p_2} Y.$$

We have $\mathrm{cl}(Y,X) \leq 3$ because we have the following sequence of spaces:

$$X_0 = X = S^3 \times Y,$$

$$X_1 = X_0 \cup_{S^3} e^4 = \left(Y \vee \left(\bigvee_{i\in I} S^{n_i+3}\right) \cup \left(\bigcup_{j\in J} e^{m_j+3}\right)\right),$$

$$X_2 = X_1 \cup_{\bigvee_{i\in I} S^{n_i+3}} \bigcup_{i\in I} e^{n_i+4} \cong Y \vee \left(\bigvee_{j\in J} S^{m_j+3}\right),$$

$$X_3 \cong Y.$$

The homology of the fibre, S^3, is finite dimensional but the Poincaré series of the loop space on the basis is not necessarily rational, as shown by Anick [2].

In the case of the adjunction of only two cells, we can prove:

Theorem 3. *Let Y be a simply connected space obtained by adjunction of two cells to a simply connected space X. If the injection $i : X \hookrightarrow Y$ is not a rational homotopy equivalence and if $H_*(\Omega Y;\mathbb{Q})$ has exponential growth, then the homology of the homotopy fibre of i has also exponential growth.*

In Theorem 3, the requirement on the number of cells is important: We have only to consider the inclusion

$$X = S_a^3 \vee S_b^3 \hookrightarrow Y = (S_a^3 \vee S_b^3) \times S_c^5.$$

Since $Y = (X \vee S_c^5) \cup_{[a,c]} e^8 \cup_{[b,c]} e^8$, we have $\mathrm{cl}(Y,X) \leq 2$. The space Y is obtained by adjunction of three cells to X. The Pontryagin algebra $H_*(\Omega Y;\mathbb{Q})$ is isomorphic to the tensor product $k[c_4] \otimes T(a_2, b_2)$ and has exponential growth. We finally remark that the homology of the homotopy fibre $F = \Omega S^5$ has only polynomial growth.

Example. Denote by X the coformal [1] space whose rational homotopy Lie algebra is defined by

$$\pi_*(\Omega X) \otimes \mathbb{Q} = \mathbb{L}(a,b)/([a,[a,b]],[b,[a,b]]),$$

[1] A space S is called coformal with rational homotopy Lie algebra L if its Sullivan minimal model [13] is isomorphic to the cochain algebra on L.

with $|a| = |b| = 2$. The space X admits a cellular decomposition of the form

$$X = S^3_a \vee S^3_b \cup_{[a,[a,b]]} e^8 \cup_{[b,[a,b]]} e^8 \cup_\omega e^{11}.$$

We consider the injection $W = S^3_a \vee S^3_b \hookrightarrow X$, and we denote by F its homotopy fibre. In this case, the reduced homology of F is in fact the quotient of the free $\mathrm{H}_*(\Omega X;k)$-module generated by two element x and y of degree 7 modulo one relation $xa - yb$.

This example is a particular case of the following more general situation:

Theorem 4. *Let X be a simply connected CW complex and $f : W \to X$ be a continuous map from a wedge of spheres into X. We suppose that $\pi_*(f)\otimes\mathbb{Q}$ is a surjective map, and is an isomorphism on the indecomposable elements. The homotopy fibre of f, F, is then rationally a suspension, and for $p \geq 0$ we have*

$$\text{dimension } \mathrm{Tor}_p^{\mathrm{H}_*(\Omega X;\mathbb{Q})}(\tilde{\mathrm{H}}_*(F;\mathbb{Q}),\mathbb{Q}) = \text{dimension } \mathrm{Tor}_{p+2}^{\mathrm{H}_*(\Omega X;\mathbb{Q})}(\mathbb{Q},\mathbb{Q}).$$

Proof. Since $\pi_*(f)\otimes\mathbb{Q}$ is surjective, the homotopy long exact sequence of the homotopy fibration $F \to W \to X$ reduces to a short exact sequence of graded Lie algebras

$$0 \to \pi_*(\Omega F)\otimes\mathbb{Q} \to \pi_*(\Omega W)\otimes\mathbb{Q} \to \pi_*(\Omega X)\otimes\mathbb{Q} \to 0.$$

Since W is a wedge of spheres, the graded Lie algebra $\pi_*(\Omega W)\otimes\mathbb{Q}$ is free. The same is therefore true for $\pi_*(\Omega F)\otimes\mathbb{Q}$ because a graded ideal in a free graded Lie algebra is also a free graded Lie algebra. We thus have

$$\pi_*(\Omega F)\otimes\mathbb{Q} = \mathbb{L}(a_i)_{i\in I}, \quad |a_i| = n_i.$$

Denote now by Z the wedge of spheres $Z = \bigvee_{i\in I} S^{n_i+1}$ and by $T \to T_0$ the rationalisation of a simply connected space T. Then the continuous map $g : Z \to F_0$ obtained by choosing for each $i \in I$ a representative $g : S^{n_i+1} \to F_0$ of the element $a_i \in \pi_{n_i}(\Omega F_0)$ is a rational homotopy equivalence.

We now consider the Hochschild–Serre spectral sequence associated with the extension of Lie algebras

$$0 \to \pi_*(\Omega F)\otimes\mathbb{Q} \to \pi_*(\Omega W)\otimes\mathbb{Q} \to L \to 0,$$

with $L = \pi_*(\Omega X)\otimes\mathbb{Q}$. As the first two terms are free Lie algebras, the differentials d^2 of the spectral sequence

$$d^2_{p,0} : \mathrm{Tor}_p^{\mathrm{UL}}(\mathbb{Q},\mathbb{Q}) \longrightarrow \mathrm{Tor}_{p-2}^{\mathrm{UL}}(\mathrm{Tor}_1^{\mathrm{H}_*(\Omega F;\mathbb{Q})}(\mathbb{Q},\mathbb{Q}),\mathbb{Q})$$

are isomorphisms for $p \geq 2$. This implies the result. □

To prove our theorems we use the Adams–Hilton model of a 1-connected CW complex as defined by [1] and the model of $\mathscr{C}_*(F)$ as a $\mathscr{C}_*(\Omega Y)$-module as defined in [9]. To be self-contained and to explain how to compute explicit examples, we recall the basic ideas of these constructions in Section 2. That section contains also the proofs of Theorem 1 and 3. The structure of n-stage differential modules and the proof of Theorem 2 are contained in Section 3.

2. Adams–Hilton models

2.1. The general framework

In spite of the fact that part of our results can be extended over arbitrary commutative rings, we have decided, for sake of simplicity, to limit ourselves to the case the coefficients are taken in a field k. Henceforth, all graded vector spaces and algebras are defined over the field k. The degree of an homogeneous element x of a graded vector space V will be denoted by $|x|$.

Let $Y = \bigcup_{\alpha\in A} e_\alpha$ be a CW complex with A a well-ordered set, and such that for each $\alpha \in A$, the union $\bigcup_{\beta\leq\alpha} e_\beta$ is a subcomplex of X. We then form the tensor algebra $(T(x_\alpha), \alpha \in A)$, with $|x_\alpha| = \dim e_\alpha - 1$. The construction of Adams and Hilton [1] provides us with a differential d on the tensor algebra $T(x_\alpha)$ and a morphism of differential graded algebras

$$(T(x_\alpha), d) \longrightarrow \mathscr{C}_*(\Omega Y; k),$$

inducing an isomorphism in homology.

The differential d is built inductively along the cellular structure of Y and is a algebrization of the attaching maps of the cells. Suppose indeed that $Y = Z \cup_\varphi e^n$ and that

$$\theta_Z : (T(V), d) \to \mathscr{C}_*(\Omega Z; k)$$

is an Adams–Hilton model for Z. The attaching map $\varphi : S^{n-1} \to Z$ induces a map in homology

$$\mathrm{H}_{n-2}(\Omega\varphi; k) : \mathrm{H}_{n-2}(\Omega S^{n-1}; k) \to \mathrm{H}_{n-2}(\Omega Z; k).$$

We denote by α the image of a generator of $\mathrm{H}_{n-2}(\Omega S^{n-1}; k)$. Since $\mathrm{H}(\theta_Z)$ is an isomorphism, there is a cycle u in $(T(V), d)$ such that $\mathrm{H}(\theta_Z)([u]) = \alpha$. We define the differential algebra $(T(V \oplus kx), d)$ as an extension of $(T(V), d)$ by putting $d(x) = u$. The map θ_Z extends then naturally to a morphism of chain algebras

$$\theta_Y : (T(V \oplus kx), d) \to \mathscr{C}_*(\Omega Y; k).$$

In fact, θ_Y induces an isomorphism in homology and is an Adams–Hilton model for Y.

A crucial step in Adams–Hilton's construction is the existence of an acyclic differential D on the tensor product $[k \oplus (\oplus_\alpha k\bar{x}_\alpha, \alpha \in A)] \otimes T(x_\alpha, \alpha \in A)$, with $|\bar{x}_\alpha| = |x_\alpha| + 1$, and

$$D(\bar{x}_\alpha \otimes 1) = 1 \otimes x_\alpha + \sum_{\beta<\alpha} \bar{x}_\beta \otimes \omega_{\alpha,\beta},$$
$$D(\bar{x}_\alpha \otimes u) = (-1)^{|\bar{x}_\alpha|}\bar{x}_\alpha \otimes d(u) + D(\bar{x}_\alpha \otimes 1)\cdot u.$$

The interest of this construction appears more clearly when we consider the homotopy fibre F of the inclusion $i : X = \bigcup_{\alpha\in B} e_\alpha \hookrightarrow Y = \bigcup_{\alpha\in A} e_\alpha$, with $B \subset A$. There exists then a morphism of right differential $T(x_\alpha)_{\alpha\in A}$-modules

$$([k \oplus (\oplus_{\alpha\in B} k\bar{x}_\alpha)] \otimes T(x_\alpha)_{\alpha\in A}, D) \longrightarrow \mathscr{C}_*(F;k)$$

inducing an isomorphism in homology [9].

2.2. cl(Y, X) and proof of Theorem 1

Suppose that $\mathrm{cl}(Y,X) = n$. Let

$$\begin{array}{ccc} (T(V),d) & \xrightarrow{\cong} & C_*(\Omega X) \\ \downarrow & & \downarrow C_*(\Omega i) \\ (A,d) = (T(V\oplus W),d) & \xrightarrow{\cong} & C_*(\Omega Y) \end{array}$$

be the Adams–Hilton diagram for the inclusion $X \hookrightarrow Y$. The cellular construction of the models implies that we have a decomposition

$$W = W_0 \oplus \ldots \oplus W_{n-1} \quad \text{with } d(W_i) \subset T(V \oplus (\oplus_{j<i} W_j)).$$

We have therefore a short exact sequence of differential right A-modules.

$$0 \to ((k\oplus sV)\otimes A, D) \to ((k \oplus sV \oplus sW)\otimes A, D) \to (sW \otimes A, \bar{D}) \to 0.$$

As $\mathrm{H}((k\oplus sV\oplus sW)\otimes A, D) \cong k$, we have an isomorphism of right $\mathrm{H}(A)$-modules

$$\tilde{\mathrm{H}}_*(F;k) \cong s^{-1}\mathrm{H}(sW\otimes A, \bar{D}).$$

The differential module $(sW \otimes A, \bar{D})$ is by construction an n-stage differential graded module. By hypothesis, $\mathrm{H}(Y,X;k) \neq 0$. Theorem 2 implies therefore Theorem 1. The proof of Theorem 2 will be given in Section 3.

2.3. The case cl(Y, X) ≤ 2 and Theorem 3

Following the lines of the previous subsection, $\tilde{\mathrm{H}}_*(F;k)$ is isomorphic to the homology of a 2-stage differential graded A-module,

$$(sW\otimes A, \bar{D}), \quad sW = sW_0 \oplus sW_1.$$

We filter $(sW\otimes A, \bar{D})$ by putting A and sW_0 in degree zero and sW_1 in degree one. This generates a spectral sequence. Since $d^0 = 1\otimes d_A$, the E^1-term is

$$(sW\otimes \mathrm{H}(A), d^1).$$

We deduce directly an extension of $\mathrm{H}(A)$-modules

$$0 \to E^2_{0,*} \to \tilde{\mathrm{H}}_*(F;k) \to E^2_{1,*} \to 0.$$

In the "two cells" case, the morphism d^1 is a morphism of H(A)-modules of the form

$$d^1 : x.\mathrm{H}(A) \to y.\mathrm{H}(A), \qquad d^1(x) = y.\omega,$$

for some cycle $\omega \in A$. Since $\mathrm{H}(Y,X;\mathbb{Q}) \neq 0$, the element ω belongs to $\mathrm{H}_+(A)$. This implies that the image of d^1 does not contain any primitive element of degree greater than $|\omega|$. On the other hand, by the Milnor–Moore theorem [12] the algebra $\mathrm{H}_*(\Omega Y;\mathbb{Q})$ is the enveloping algebra of the Lie algebra $\pi_*(\Omega Y)\otimes\mathbb{Q}$ and this Lie algebra has also an exponential growth. Therefore, Coker d^1 contains a graded vector space which has an exponential growth. □

3. n-stage differential A-modules

Let (A,d_A) be a k-augmented chain algebra over a field k and let $(W_*\otimes A,D)$ be a semifree A-module. Since the d_0-part of the differential is $1\otimes d_A$, the filtration of $(W_*\otimes A,D)$ by the submodules $(W_{\leq n}\otimes A,D)$ yields a canonical spectral sequence whose term (E^1,d^1) has the form $(W_*\otimes H(A),\delta)$ with $\delta(W_p)\subset W_{p-1}\otimes H(A)$ for $p\geq 1$.

Definition. The semifree A-module $(W_*\otimes A,D)$ is called a *strong* semifree A-module if $H_p(E^1,\delta^1)=0$ for $p>0$.

Strong semifree modules are called distinguished resolutions in Gugenheim–May [10]. An important property of strong semifree modules is the following result of Gugenheim and May:

Lemma 1 (Gugenheim et al. [10, Theorem 1.2]). *For each differential A-module (C,d) there is a quasi-isomorphism of differential A-modules*

$$\varphi : (W_*\otimes A,D)\to(C,d),$$

where $(W_*\otimes A,D)$ is a strong semifree A-module.

Lemma 2. *Let $(C,d)=(V_*\otimes A,d)$ be a semifree A-module and let $(W_*\otimes A,D)$ be a strong semifree A-module. Then each morphism of differential A-modules $f:(C,d)\to(W_*\otimes A,D)$ is homotopic to a morphism g satisfying the property $g(V_r)\subset W_{\leq r}\otimes A$ for $r\geq 0$.*

Proof. We decompose the differential D into the form $D=d_0+d_1+\cdots$ with $d_p(W_r)\subset W_{r-p}\otimes A$. The space V_0 is a vector space consisting of cycles. We choose a graded basis $\{x_i\}$ of V_0 and we write $f(x_i)$ in the form

$$f(x_i)=f_0(x_i)+\cdots+f_{n_i}(x_i), \quad f_j(x_i)\in W_j\otimes A.$$

Suppose $n_i > 0$. In that case $d_0 f_{n_i}(x_i) = 0$ and $d_1 f_{n_i}(x_i) = -d_0 f_{n_i-1}(x_i)$. This shows that $f_{n_i}(x_i)$ defines a class α_i in $W_{n_i} \otimes \mathrm{H}(A) = E^1_{n_i}$ and that $\delta^1(\alpha_i) = 0$. Since $(W_* \otimes A, D)$ is a strong semifree module, it follows that

$$\alpha_i = \delta^1(\beta_i), \qquad \beta_i \in W_{n_i+1} \otimes \mathrm{H}(A).$$

Represent β_i by a d_0-cycle $u_i \in W_{n_i+1} \otimes A$. There exists then an element $v_i \in W_{n_i} \otimes A$ with $d_1(u_i) + d_0(v_i) = f_{n_i}(x_i)$.

The morphism of differential A-modules $H : V_* \otimes A \to W_* \otimes A$ defined by $H(V_r \otimes A) = 0$ for $r \neq 0$ and $H(x_i) = u_i + v_i$ on V_0 defines an homotopy between f and a map f' satisfying $f'(x_i) \in W_{<n_i} \otimes A$. An iterated induction applied to each element of the basis $\{x_i\}$ constructs an homotopy between f and a map g with $g(V_0) \subset W_0 \otimes A$.

By induction we suppose henceforth that $f(V_j) \subset W_{\leq j} \otimes A$ for $j = 0, \ldots, r-1$, and we take a basis $\{x_i\}$ of V_r. Once again we decompose $f(x_i)$ as

$$f(x_i) = f_0(x_i) + \cdots + f_{n_i}(x_i), \qquad f_j(x_i) \in W_j \otimes A.$$

Suppose $n_i > r$, then $d_0 f_{n_i}(x_i) = 0$ and $d_1 f_{n_i}(x_i) = -d_0 f_{n_i-1}(x_i)$. The same argument as before enables then to replace f by an homotopic map f' satisfying $f'(V_r) \subset W_{\leq r} \otimes A$. □

Lemma 3. *Let $(R_{<n} \otimes A, D)$ be an n-stage differential module, let $(C, D) = (W_* \otimes A, D)$ be a strong semifree A-module and let $g : (W_* \otimes A, D) \to (R_* \otimes A, D)$ be a morphism of differential A-modules. Suppose the grade of* H(C) *is greater than or equal to m. Then g is homotopic to a map g' satisfying $g'(W_r) \subset R_{<r+n-m} \otimes A$, for $r \geq 0$.*

Proof. To prove the existence of the map g', we first remark that $g(W_r) \subset R_{\leq r+n-1} \otimes A$ for all $r \geq 0$. We denote by p the smallest integer such that g is homotopic to a map θ with $\theta(W_r) \subset R_{\leq r+p} \otimes A$ for $r = 0, \ldots$. If $p \leq n - m - 1$, the theorem is proved. We suppose therefore $p > n - m - 1$.

We write $\theta = \theta_p + \theta_{p-1} \ldots$ with $\theta_q(W_r) \subset R_{r+q} \otimes A$, and we denote by s the smallest integer r such that there exists a map θ homotopic to g with

$$\theta(W_r) \subset R_{\leq r+p} \otimes A \quad \text{for } r = 0, \ldots,$$
$$\theta_p(W_r) = 0 \quad \text{for } r > s.$$

We want to arrive at a contradiction. We decompose the differentials into the form $D = d_0 + d_1 + \cdots$ with $d_t(R_q) \subset R_{q-t} \otimes A$ and $d_t(W_q) \subset W_{q-t} \otimes A$ for $t = 1, \ldots$.

Since $\theta_p \circ d_0 = d_0 \circ \theta_p$, the map θ induces a morphism of H(A)-modules $\rho : W_s \otimes \mathrm{H}(A) \to R_{s+p} \otimes \mathrm{H}(A)$. Now since $\theta D = D\theta$ and $\theta_p(W_{s+1}) = 0$, we have $\theta_p d_1 = d_0 \theta_{p-1}$ on W_{s+1} and therefore $\rho \circ \delta^1 = [\theta_p \circ d_1] = 0$. This means that ρ defines a cocycle of degree s in the complex $\mathrm{Hom}_{\mathrm{H}(A)}(W_* \otimes \mathrm{H}(A), R_{s+p} \otimes \mathrm{H}(A))$. Remark that the homology in degree s of this complex is $\mathrm{Ext}^s_{\mathrm{H}(A)}(\mathrm{H}(C), R_{s+p} \otimes \mathrm{H}(A))$. Since $\theta_p(W_s) \neq 0$, $R_{s+p} \neq 0$, so that $s + p \leq n - 1$ and since $p > n - m - 1$ we get $s < m$. Since the grade of H(C) is $\geq m$ the element ρ is a coboundary : This means that there is an H(A)-linear map $\psi : W_{s-1} \otimes \mathrm{H}(A) \to R_{s+p} \otimes \mathrm{H}(A)$ such that $\rho = \psi \circ \delta^1$. Denote by $\{x_i\}$ and $\{y_j\}$ graded bases, respectively, for W_s and for W_{s-1}. Denote also by $\{\beta_j\}$ a family of d_0-cycles in $R_{s+p} \otimes A$ such that $\psi(y_j) = [\beta_j]$. The correspondence $y_j \rightsquigarrow \beta_j$ extends to an A-linear map $\Psi : W_{s-1} \otimes A \to R_{s+p} \otimes A$. For each element x_i the element $\theta_p(x_i) - \Psi \circ d_1(x_i)$ is therefore a d_0-boundary,

$$\theta_p(x_i) - \Psi \circ d_1(x_i) = d_0(z_i).$$

We introduce then the A-linear map $h : W_* \otimes A \to R_{<n} \otimes A$,

$$\begin{aligned} &h(W_q) = 0 \quad \text{for } q \neq s, s-1, \\ &h = \Psi \quad \text{on } W_{s-1} \otimes A, \\ &h(x_i) = (-1)^{|h|+1} z_i \quad \text{on } W_s. \end{aligned}$$

The map $\theta' = \theta - (hD - (-1)^{|h|} Dh)$ is homotopic to θ and satisfies $\theta'(W_r) \subset W_{\leq r+p} \otimes A$ for all r and $\theta'(W_r) \subset R_{\leq r+p-1} \otimes A$ for $r \geq s$. This is in contradiction with our hypothesis on p and s. □

Corollary. *Let $(C, D) = (W_* \otimes A, D)$ be a strong semifree A-module and let $g : (W_* \otimes A, D) \to (A, d_A)$ be a morphism of differential A-modules. Suppose the grade of* H(C) *is greater than or equal to m. Then g is homotopic to a map g' that sends $W_{<m}$ to zero : $g'(W_{<m}) = 0$.*

We now proceed to the proof of Theorem 2.

Theorem 2. *Let (C, d) be an n-stage differential A-module. If* $\mathrm{H}(C) \neq 0$, *then* $\mathrm{grade}_{\mathrm{H}(A)}\mathrm{H}(C) < n$. *Moreover, if* $\mathrm{grade}_{\mathrm{H}(A)}\mathrm{H}(C) = n - 1$, *then* H$(C)$ *is an* H(A)-*module of homological dimension $n - 1$.*

Proof. Let $F : (W_* \otimes A, D) \to (C, d)$ be a quasi-isomorphism as given by Lemma 1. Since (C, d) is semi-free, by Proposition 2, F admits an inverse up to homotopy $G : (C, d) \to (W_* \otimes A, D)$. By Lemma 2, we can even choose G in such a way that $G = i \circ G'$,

$$(C, d) \xrightarrow{G'} (W_{<n} \otimes A, D) \xrightarrow{i} (W_* \otimes A, D).$$

The maps

$$(W_* \otimes A, D) \xrightarrow{G' \circ F} (W_{<n} \otimes A, D) \overset{i}{\hookrightarrow} (W_* \otimes A, D)$$

show that $(W_* \otimes A, D)$ is an homotopy retract of $(W_{<n} \otimes A, D)$.

If the grade of H(C) is greater than or equal to n, than by Lemma 3, $G' \circ F$ is homotopic to a map θ satisfying $\theta(W_r) \subset W_{\leq r-1} \otimes A$ for $r = 0, \ldots$. In particular $\theta(W_0) = 0$. This means that each element in W_0 is a boundary, so that $\mathrm{H}(C) = 0$.

If the grade of $\mathrm{H}(C) = n - 1$, then by Lemma 3, $G' \circ F$ is homotopic to a map ψ such that $\psi(W_r) \subset W_{\leq r} \otimes A$. In that case we filter $W_* \otimes A$ and $W_{<n} \otimes A$ by the filtration

degree in W_*. The map ψ preserves the filtrations, and therefore $(W_* \otimes \mathrm{H}(A), \delta^1) = E^1(W_* \otimes A, D)$ is an homotopy retract of $(W_{<n} \otimes \mathrm{H}(A), \delta^1) = E^1(W_{<n} \otimes A, D)$:

$$(W_* \otimes \mathrm{H}(A), \delta^1) \xrightarrow{E^1(\psi)} (W_{<n} \otimes \mathrm{H}(A), \delta^1) \overset{j}{\hookrightarrow} (W_* \otimes \mathrm{H}(A), \delta^1).$$

Denote by $h : W_* \otimes \mathrm{H}(A) \to W_* \otimes \mathrm{H}(A)$ the homotopy between 1 and $j \circ E^1(\psi)$,

$$1 - jE^1(\psi) = \delta^1 h + h\delta^1.$$

We form now the complex $(W_*, \bar{\delta}^1) = ((W_* \otimes \mathrm{H}(A)) \otimes_{\mathrm{H}(A)} k, \bar{\delta}^1)$, and we denote by $\bar{h}$ the homotopy induced by h. Since $E^1(\psi)(W_{\geq n}) = 0$, we have $1 = \bar{h}\bar{\delta}^1 + \bar{\delta}^1\bar{h}$ on $W_{\geq n}$. This implies that $\mathrm{Tor}_p^{\mathrm{H}(A)}(\mathrm{H}(C), k) = \mathrm{H}_p(W_*) = 0$ for $p \geq n$. Therefore, the homological dimension of $\mathrm{H}(C)$ is less than n. The homological dimension then has to be equal to $n - 1$ because the grade of a module is always less than or equal to its homological dimension. □

Let $(C, d) = (V_* \otimes A, d)$ be an n-stage differential A-module. The projection ψ: $(C, d) \to (V_{n-1} \otimes A, 1 \otimes d_A)$ is then a morphism of differential A-modules. If ψ is not homotopically trivial, then ψ induces a nontrivial map from (C, d) into (A, d). The map ψ could be homotopically trivial, but by the next proposition, when A is connected, there always exists a nontrivial morphism of (C, d) into (A, d).

An example of such a situation happens in topology when we consider a relative n-cone (Y, X), i.e. when we have a sequence of cofibrations

$$E_i \xrightarrow{f_i} X_i \longrightarrow X_{i+1} = C_{f_i}, \quad i = 0, \ldots, n-1,$$

with $X_0 = X$, $X_n = Y$, and where the spaces E_i are wedges of spheres. In this situation we consider the sequence of inclusions $X \overset{i}{\hookrightarrow} X_{n-1} \overset{j}{\hookrightarrow} Y$ and the homotopy fibration obtained by taking the homotopy fibres of the inclusions i, ji and j:

$$F_i \to F_{ji} \xrightarrow{p} F_j.$$

The morphism p is a morphism of $\mathscr{C}_*(\Omega Y; k)$-modules from an n-stage differential module to a free module. Remark at this point that if the grade of $\mathrm{H}_*(F_{ji}; k)$ is positive then $\tilde{\mathrm{H}}_*(p; k) = 0$.

The results in this text concern homotopy classes of maps between A-modules. A very interesting existence theorem is given by Proposition 3 below.

Proposition 3. *Let (C, d) be an n-stage differential module over a connected differential graded algebra (A, d_A). If $\mathrm{H}(C) \neq 0$, then there is a surjective map of A-modules $(C, d) \to (A, d_A)$ which is not homotopically trivial.*

Proof. We decompose $V = X \oplus \mathrm{Im}\, \bar{d}$, where $\bar{d}$ is the linear part of the differential d, $\bar{d} : V \to V$.

For differential A-modules (M, d) and (N, d), we will consider the complex

$$(\mathrm{Hom}_A(M, N), \Delta)$$

where $\mathrm{Hom}_A(M,N)$ denotes the graded vector space of A-linear maps from M into N and where $\Delta(f) = df - (-1)^{|f|} fd$.

Every linear form $\varepsilon : V_{n-1} \to k$ extends to a cocycle φ in $\mathrm{Hom}_A(C,A)$ by putting $\varphi(x) = \varepsilon(x)$ for $x \in V_{n-1}$ and $\varphi(x) = 0$ for $x \in V_{<n-1}$. If one of these cocycles is not a coboundary, then we have found a non-homotopically trivial surjective morphism $\varphi : C \to A$. Otherwise, by a decreasing induction on r we suppose that

(1) for every linear form $\varepsilon : X_s \to k$, $r < s < n$, there is a cocycle $\varphi \in \mathrm{Hom}_A(C,A)$ such that $(\varphi - \varepsilon)(X) \subset A_+$,

(2) all such cocyles are coboundaries,

and we will construct for every linear form $\varepsilon : X_r \to k$ a cocyle φ in $\mathrm{Hom}_A(C,A)$ with $(\varphi - \varepsilon)(X) \subset A_+$.

All the cocycles associated with linear forms cannot be coboundaries for all r, because in this case we may deduce that $\bar{d}$ is injective on X, so that $\mathrm{H}_*(V,\bar{d}) = 0$, and $\mathrm{H}_*(C,d) = 0$, which is not possible.

We now proceed to the inductive step of the proof. Let $\varepsilon : X_r \to k$ be a linear form. This defines a cocycle $\varphi \in \mathrm{Hom}_A(V_{\leq r} \otimes A, A)$ by putting $\varphi(x) = \varepsilon(x)$ for $x \in V_r$ and $\varphi(x) = 0$ for $x \in V_{<r}$. We suppose to have extended φ into a cocycle $\psi \in \mathrm{Hom}_A(V_{\leq s} \otimes A, A)$ with the property $(\psi - \varepsilon)(X) \subset A_+$ and we consider the extension to $V_{\leq s+1} \otimes A$ defined by $\psi(V_{s+1}) = 0$. Of course, $\Delta(\psi) = 0$ on $V_{\leq s}$, and $\Delta(\psi)$ maps V_{s+1} into the vector space $Z(A)$ consisting of the cycles in A. We write

$$\Delta(\psi) = \sum_i \varepsilon_i \cdot \alpha_i,$$

where the ε_i are linear forms on V_{s+1} and the α_i a graded basis of $Z(A)$. Extending the linear forms ε_i by zero on $V_{\leq s}$ we obtain cocycles φ_i in $\mathrm{Hom}_A(V_{\leq s+1} \otimes A, A)$ that are coboundaries

$$\varphi_i = \Delta(\beta_i).$$

The morphism $\psi_1 = \psi - \sum_i \beta_i \cdot \alpha_i$ is then a cocyle with the required properties. □

Acknowledgements

The author would like to thank Pascal Lambrechts and Clas Löfwall for helpful comments and suggestions.

References

[1] J.F. Adams and P. Hilton, On the chain algebra of a loop space, Comment. Math. Helv. 30 (1955) 305–330.

[2] D. Anick, A counterexample to a conjecture of Serre, Ann. Math. 115 (1982) 1–33.

[3] O. Cornea, Cone-length and Lusternik-Schnirelmann category, Topology 33 (1984) 95–111.

[4] Y. Félix and S. Halperin, Rational L.S. category and its applications, Trans. Amer. Math. Soc. 273 (1982) 1–37.

[5] Y. Félix, S. Halperin and J.-C. Thomas, Gorenstein spaces, Adv. Math. 71 (1988) 92–112.
[6] Y. Félix, S. Halperin and J.-C. Thomas, Loop space homology of spaces of LS category one and two, Math. Ann. 287 (1990) 377–386.
[7] Y. Félix, S. Halperin and J.-C. Thomas, The category of a map and the grade of a module, Israel J. Math. 78 (1992) 177–196.
[8] Y. Félix, S. Halperin and J.-C. Thomas, in: I. James, ed., Differential algebras in topology. Handbook of Algebraic Topology (North-Holland, Amsterdam, 1995) 829–866.
[9] Y. Félix and J.-C. Thomas, Module d'holonomie d'une fibration, Bull. Soc. Math. France 113 (1985), 255-258.
[10] V. Gugenheim and P. May, On the theory and applications of differential torsion products, Mem. AMS 142 (1974).
[11] S. Halperin and J. Stasheff, Obstructions in homotopy equivalences, Adv. Math. 32 (1979) 233–279.
[12] J. Milnor and J.C. Moore, On the structure of Hopf algebras, Ann. Math. 81 (1965) 211–264.
[13] D. Sullivan, Infinitesimal computations in topology, Publ. Math. IHES 47 (1977) 269–331.